STABILITY OF THE CIRCULAR COUETTE FLOW OF A POWER-LAW FLUID

V. S. Belokon'

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The stability of the circular Couette flow of a non-Newtonian power-law fluid is analyzed in the small-gap approximation.

We consider the stability of a steady laminar flow in a gap between two infinitely long coaxial cylinders, the fluid being incompressible and its viscosity being a function of the strain rate. For such a fluid the relation between the stress tensor components and the strain tensor components can be expressed as follows:

$$\sigma_{ij} = -p\delta_{ij} + 2\mu(H)e_{ij} \quad (i,\ j = 1,\ 2,\ 3). \tag{1}$$

Here the viscosity μ is a function of the intensity of shearing strains $H = (2e_{ij}e_{ji})^{1/2}$ and δ_{ij} is the Kronecker delta.

We express the pressure as well as the strain tensor components as sums

$$p = p^0 + p', \quad e_{ij} = e_{ij}^0 + e_{ij}', \tag{2}$$

where the first terms represent the steady motion and the second terms represent the perturbation field.

We will now consider the motion of the fluid in a cylindrical system of coordinates, with the axes 1, 2, 3 denoting, respectively, the axes φ, r, z. For the particular type of flow under consideration we have

$$e_{ij}^0 = 1/2\dot{\gamma}_0\delta_{i1}\delta_{j2}, \tag{3}$$

with $\dot{\gamma}_0$ denoting the shear rate in an unperturbed flow.

With each component of the strain tensor in the expression for H written in terms of (2) and with the e_{ij}' terms assumed small, we obtain, with the aid of (3), the following expression for the viscosity of the fluid in (1):

$$\mu(H) = \mu(\dot{\gamma}_0) + 2e_{12}'\partial\mu/\partial\dot{\gamma}_0, \tag{4}$$

where the symbol $\partial\mu/\partial\dot{\gamma}_0$ signifies a derivative at the point $\dot{\gamma} = \dot{\gamma}_0$. The components of the stress tensor become then

$$\sigma_{ij} = -(p^0 + p')\delta_{ij} + [\mu(\dot{\gamma}_0) + 2e_{12}'\partial\mu/\partial\dot{\gamma}_0](\dot{\gamma}_0\delta_{i1}\delta_{j2} + 2e_{ij}'),$$

from where we obtain an expression for the perturbations of the stress tensor components:

$$\sigma_{ij}' = -p'\delta_{ij} + 2e_{12}'\dot{\gamma}_0\partial\mu/\partial\dot{\gamma}_0\delta_{i1}\delta_{j2} + 2\mu(\dot{\gamma}_0)e_{ij}'.$$

Since in this type of steady circular flow the trajectories of all particles constitute arcs of concentric circles, hence the components of the unperturbed flow velocity v_r^0 and v_z^0 are zero. From the system of equations for a steady unperturbed flow we deduce that here (see [1], e.g.) the tangential velocity component v_φ^0 is invariant with respect to the coordinates φ and z. If in this case we assume also that the perturbation field is invariant with respect to φ, then we obtain, in the conventional manner, the following equations for the perturbations superposed on the principal circular motion:

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$$\frac{\partial v_r'}{\partial t}-\frac{2v_\varphi^0 v_\varphi'}{r}=-\frac{1}{\rho}\frac{\partial p'}{\partial r}+\frac{\mu_0}{\rho}\left(\Delta_1 v_r'-\frac{v_r'}{r^2}\right)+\frac{2}{\rho}\frac{\partial\mu_0}{\partial r}\frac{\partial v_r'}{\partial r}, \tag{5}$$

$$\frac{\partial v_\varphi'}{\partial t}+v_r'\left(\frac{\partial v_\varphi^0}{\partial r}+\frac{v_\varphi^0}{r}\right)=\frac{\mu_0}{\rho}\left(\Delta_1 v_\varphi'-\frac{v_\varphi'}{r^2}\right)$$

$$+\frac{\dot\gamma_0}{\rho}\frac{\partial\mu_0}{\partial\dot\gamma_0}\left(\frac{\partial^2 v_\varphi'}{\partial r^2}+\frac{1}{r}\frac{\partial v_\varphi'}{\partial r}-\frac{v_\varphi'}{r^2}\right)+\frac{r}{\rho}\frac{\partial}{\partial r}\left(\frac{v_\varphi'}{r}\right)\frac{\partial}{\partial r}\left(\dot\gamma_0\frac{\partial\mu_0}{\partial\dot\gamma_0}+\mu_0\right), \tag{6}$$

$$\frac{\partial v_z'}{\partial t}=-\frac{1}{\rho}\frac{\partial p'}{\partial z}+\frac{\mu_0}{\rho}\Delta_1 v_z'+\frac{1}{\rho}\left(\frac{\partial v_z'}{\partial r}+\frac{\partial v_r'}{\partial z}\right)\frac{\partial\mu_0}{\partial r}, \tag{7}$$

$$\frac{\partial(rv_r')}{\partial r}+\frac{\partial(rv_z')}{\partial z}=0,\ \Delta_1\equiv\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2} \tag{8}$$

with the boundary values

$$v_r'=v_\varphi'=v_z'=0 \quad \text{at} \quad r=R_1 \text{ and } r=R_2. \tag{9}$$

In Eqs. (5)-(8) the viscosity of the fluid is $\mu_0=\mu[\dot\gamma_0(r)]$, the density of the fluid is ρ, and the velocity components of the perturbed flow are v_r', v_φ', v_z'.

We will seek the perturbation field in the form

$$v_r'=\exp(\beta t)\,u_1(r)\cos\lambda z,\quad v_\varphi'=\exp(\beta t)\,u_2(r)\cos\lambda z,\quad v_z'=\exp(\beta t)\,u_3(r)\sin\lambda z. \tag{10}$$

Inserting (10) into Eqs. (5)-(8) and eliminating p', we obtain

$$\left[\left(1+\frac{\dot\gamma_0}{\mu_0}\frac{\partial\mu_0}{\partial\dot\gamma_0}\right)L-\lambda^2-\frac{\beta\rho}{\mu_0}\right]u_2+\left(\frac{\partial u_2}{\partial r}-\frac{u_2}{r}\right)\frac{1}{\mu_0}\frac{\partial}{\partial r}\left(\dot\gamma_0\frac{\partial\mu_0}{\partial\dot\gamma_0}+\mu_0\right)=\frac{\rho}{\mu_0}u_1\left(\frac{\partial v_\varphi^0}{\partial r}+\frac{v_\varphi^0}{r}\right), \tag{11}$$

$$\left(L-\lambda^2-\frac{\beta\rho}{\mu_0}\right)(L-\lambda^2)u_1+\frac{1}{\mu_0}\frac{\partial^2\mu_0}{\partial r^2}(L+\lambda^2)u_1$$

$$+\frac{1}{\mu_0}\frac{\partial\mu_0}{\partial r}\cdot\frac{\partial}{\partial r}(L-\lambda^2)u_1=\frac{2\rho}{\mu_0}\frac{\lambda^2}{r}v_\varphi^0 u_2,\ L\equiv\frac{\partial^2}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial}{\partial r}-\frac{1}{r^2}. \tag{12}$$

With incompressibility stipulated here, the boundary conditions (9) become

$$u_1=u_2=\frac{du_1}{dr}=0 \quad \text{at} \quad r=R_1 \text{ and } r=R_2.$$

For further calculations one must know the relation $\mu_0=\mu[\dot\gamma_0(r)]$. For illustration, let us consider the flow of a so-called power-law fluid, i.e., a non-Newtonian fluid whose shear rate and shearing stress in the given type of flow are related as follows:

$$\tau=K\dot\gamma_0^n. \tag{13}$$

Here K and n are the rheological parameters of the fluid.

For the given unperturbed circular flow we have

$$\dot\gamma_0=r\frac{\partial}{\partial r}\left(\frac{v_\varphi^0}{r}\right), \tag{14}$$

and the tangential stress $\tau=Cr^{-2}$. Substituting these values for $\dot\gamma_0$ and τ into (13), we obtain an equation for v_φ^0 and, after integration,

$$v_\varphi^0=Ar+Br^{1-2/n}. \tag{15}$$

Constants A and B are determined from the boundary conditions

$$v_\varphi^0(R_1)=\Omega_1 R_1,\quad v_\varphi^0(R_2)=\Omega_2 R_2,$$

where Ω_1 and Ω_2 are the angular velocity of the inner and of the outer cylinder, respectively:

$$A=\Omega_1(\omega-\eta^{2/n})(1-\eta^{2/n})^{-1},\quad B=\Omega_1 R_1^{2/n}(1-\omega)(1-\eta^{2/n})^{-1}. \tag{16}$$

Here $\omega=\Omega_1/\Omega_2$ and $\eta=R_1/R_2$.

The effective viscosity of a power-law fluid obeying the rheological equation (13) is $\mu_0 = K\dot{\gamma}_0^{n-1}$.

Having calculated the shear rate according to (14) and taking into account (15) and (16), we obtain an expression for the viscosity as a function of the radial coordinate:

$$\mu_0 = K\,(2n|B|)^{n-1} r^{2/n-2}.$$

Many experiments dealing with the stability of the Couette flow of a viscous fluid (e.g., [2, 3]) have shown that, when this flow becomes unstable, a steady secondary flow is induced. We will assume that this happens also in the case of a fluid whose viscosity is a function of the strain rate. In order to establish the criteria of stability loss, it is sufficient to consider the case of neutral stability corresponding to $\beta = 0$.

We then have the following system of equations for a power-law fluid:

$$(L-\lambda^2)^2 u_1 + \frac{2(n-1)(3n-2)}{r^2 n^2}(L+\lambda^2)u_1 + \frac{2(1-n)}{rn}\frac{d}{dr}(L-\lambda^2)u_1 = \frac{2\lambda^2 r^{2-2/n}}{\nu_n}\frac{v_\varphi^0}{r}u_2, \tag{17}$$

$$(nL-\lambda^2)u_2 + \frac{2(1-n)}{r}\frac{du_2}{dr} + \frac{2(n-1)}{r^2}u_2 = \frac{r^{2-2/n}}{\nu_n}\left(\frac{dv_\varphi^0}{dr} + \frac{v_\varphi^0}{r}\right)u_1. \tag{18}$$

Here $\nu_n = (2n|B|)^{n-1}K\rho^{-1}$.

In Eqs. (17) and (18) we now introduce a new variable, namely

$$\zeta = (r^{1/n} - R_1^{1/n})(R_2^{1/n} - R_1^{1/n})^{-1},\ 0 \leqslant \zeta \leqslant 1$$

and consider the problem for the case where $(R_2^{1/n} - R_1^{1/n})nR_1^{-1/n} \ll 1$ (the case of a small gap). Within an accuracy down to terms of the order of $(R_2^{1/n} - R_2^{1/n})nR_1^{-1/n}$, we have then

$$\frac{v_\varphi^0}{r} = \Omega_1[1-(1-\omega)\zeta],\quad \frac{dv_\varphi^0}{dr} + \frac{v_\varphi^0}{r} = \frac{2A}{n}$$

and the system of Eqs. (17)-(18) becomes

$$\left(\frac{d^2}{d\zeta^2} - a^2\right)^2 u = (1+\alpha\zeta)v, \tag{19}$$

$$\left(n\frac{d^2}{d\zeta^2} - a^2\right)v = -\mathrm{T}a^2 u, \tag{20}$$

where

$$a^2 = \lambda^2 n^2 (R_2^{1/n} - R_1^{1/n})^2 R_1^{2-2/n};$$

$$\mathrm{T} = -4An^3\nu_n^2\Omega_1(R_2^{1/n} - R_1^{1/n})^4 R_1^{8-8/n};$$

$$u = 1/2u_1\nu_n R_1^{4/n-4}[a^2 n^2\Omega_1(R_2^{1/n} - R_1^{1/n})^2]^{-1};$$

$$v = u_2;\ \alpha = -(1-\omega).$$

Here T is the universal Taylor number.

When n = 1, Eqs. (19) and (20) become the respective equations for a Newtonian fluid. This case has been thoroughly analyzed in [4].

Thus, in order to establish the stability criteria for the circular Couette flow of a non-Newtonian power-law fluid in a small gap, it is necessary to consider the solution to the system of Eqs. (19)-(20) which will satisfy the boundary conditions

$$u = v = du/d\zeta = 0 \quad \text{at} \quad \zeta = 0 \ \text{and}\ \zeta = 1.$$

According to Chandrasechar [4], we expand v into a series:

$$v = \sum_{m=1}^{\infty} c_m \sin m\pi\zeta$$

and from (19) we find the solution for u which will satisfy the stipulated boundary conditions. We then insert u and v into (20), multiply by $k\pi\zeta$ (k = 1, 2, 3, . . .), and, integrating with respect to ζ over its range of variation, we obtain an infinite system of homogeneous linear equations, with an infinite number of un-

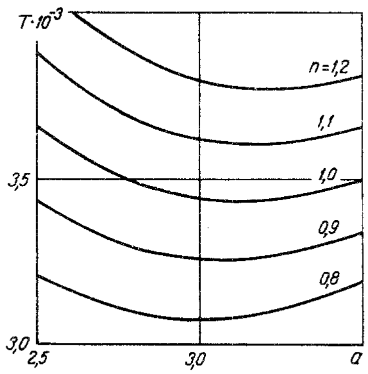

Fig. 1

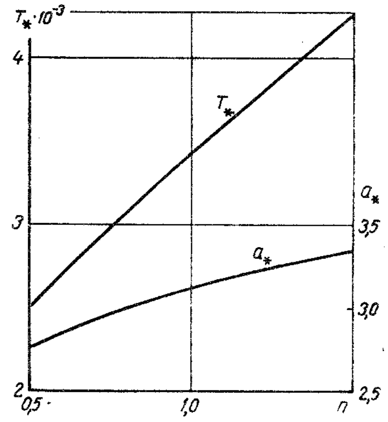

Fig. 2

Fig. 1. Universal Taylor number as a function of n and a.

Fig. 2. Critical Taylor number T_* and critical referred wave number a_* as functions of the exponent n.

known constants $c_m(m^2\pi^2 + a^2)^{-2}$. In the stipulated approximation (m = k finite) we obtain a shorter finite system of equations. In order for this system to have a nontrivial solution, its determinant Δ must be equal to zero. From this requirement we derive a relation between T, a, n, and α:

$$\Delta = \|A_{km}\| = 0,$$

$$A_{km} = 4mk\pi^2\alpha\,[(-1)^{m+k} - 1][(k^2\pi^2 + a^2)(m^2\pi^2 + a^2)]^{-1}$$
$$- 2amk\pi^2\,[(k^2\pi^2 + a^2)^2(\mathrm{sh}^2 a - a^2)]^{-1}\{(\mathrm{sh}\,a\,\mathrm{ch}\,a - a)[1 + (1 + \alpha)(-1)^{m+k}]$$
$$+ (\mathrm{sh}\,a - a\,\mathrm{ch}\,a)[(-1)^{k+1} + (1 + \alpha)(-1)^{m+1}] - a(1 + \alpha)(-1)^{m+k}$$
$$- 4a\alpha\,\mathrm{sh}\,a\,(m^2\pi^2 + a^2)^{-1}[\mathrm{sh}\,a + a(-1)^{m+1}][(-1)^{m+1} - 1]\}$$
$$+ 1/2\delta_{km} + \alpha\Psi_{km} - 1/2\,(nk^2\pi^2 + a^2)(k^2\pi^2 + a^2)^2\delta_{km}a^{-2}\mathrm{T}^{-1}. \qquad (21)$$

Here

$$\Psi_{km} = \begin{cases} 0, & \text{if } m + k \text{ is even, and } m \neq k, \quad 1/4, \quad \text{if } m = k, \\ \dfrac{4mk}{k^2 - m^2}\left[\dfrac{2}{m^2\pi^2 + a^2} - \dfrac{1}{\pi^2(k^2 - m^2)}\right], & \text{if } m + k \text{ is odd} \end{cases}$$

If (21) determines the relation T = T(a, n, α), then the critical value of the Taylor number T_* corresponding to the onset of instability is the minimum value of T(a) for each n and α. Numerical evaluation has shown that T_* calculated to the first approximation (m = k = 1) differs by about 1% from T_* calculated to the second or even the third approximation. For n = 1, the value T_* here differs by about the same amount from the value obtained in [2].

For the first approximation we obtain from (21):

$$\mathrm{T} = \frac{2(n\pi^2 + a^2)(\pi^2 + a^2)^2}{a^2(2 + \alpha)}\left\{1 - \frac{16a\pi^2\mathrm{ch}^2(a/2)}{(\pi^2 + a^2)^2(\mathrm{sh}\,a + a)}\left[1 - \frac{a(1 + \alpha)}{2(2 + \alpha)(\mathrm{sh}\,a - a)\,\mathrm{ch}^2(a/2)}\right]\right\}^{-1}. \qquad (22)$$

In Fig. 1 are shown T = T(a) curves computed according to (22) for several values of n with $\alpha = -1$ (inner cylinder rotating, outer cylinder stationary).

The critical Taylor number and the referred wave number are shown in Fig. 2 as functions of n.

Since the effective viscosity of a power-law fluid in the given type of flow varies throughout the gap, hence, in the case of a rotating inner cylinder and a stationary outer cylinder, the stability of such a flow should be determined by the fluid layer nearer to the rotating cylinder surface. For example, at n < 1 the effective viscosity becomes minimum at the surface of the inner cylinder and, evidently, the flow of such a fluid should become unstable earlier than the flow of a constant-viscosity fluid, which is also confirmed by our calculations. It follows from the preceding analysis that the angular velocity of the inner cylinder at which the flow becomes unstable is determined not only by the effective viscosity, referred to the given shear rate in the gap, but also by the change in the critical Taylor number.

NOTATION

σ_{ij}	are the components of the stress tensor;
e_{ij}	are the components of the strain tensor;
p	is the pressure;
$\dot{\gamma}$	is the shear rate;
H	is the intensity of shearing strains;
μ_0	is the effective viscosity of the fluid;
ρ	is the density of the fluid;
v_r, v_z, v_φ	are the velocity components along the respective coordinate axes;
K, n	are the rheological parameters of the fluid;
R_1, R_2	are the radius of the inner and of the outer cylinder, respectively;
Ω_1, Ω_2	are the angular velocity of the inner and of the outer cylinder, respectively;
T	is the universal Taylor number;
T_*	is the critical value of the universal Taylor number;
a_*	is the critical value of the referred wave number.

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